JUNIOR QUALIFYING EXAMINATION – August 2021

PART I

AM 1. Let $a > -1$ be a real constant. Show that $(1 + a)^n \geq 1 + na$ for all integers $n \geq 0$.

AM 2. Let $C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$.

(i) Show that the equation $C(x) = 0$ has at least one solution in the interval $[0; 2]$.

(ii) Show that the equation $C(x) = 0$ has exactly one solution in the interval $[0; 2]$.

AM 3. How many numbers are there in the set $S = \{1; 2; \ldots; 3000\}$ that are divisible by at least one of 2, 3, or 5?

AM 4. For $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$; find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

AM 5. Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$f(x; y) = \begin{cases} \frac{2x^2y}{x^4 + y^2} & \text{if } (x; y) \neq (0; 0), \\ 0 & \text{if } (x; y) = (0; 0). \end{cases}$$

(i) Is f continuous at $(0; 0)$? Explain.

(ii) Do the first partial derivatives of f exist at $(0; 0)$? If so, what are they (explain), and if not, why not?

(iii) Is f differentiable at $(0; 0)$? Explain.

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PART II

PM 1. For each of the following, either find the limit or prove divergence:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{4^n}$

(iii) $\displaystyle\lim_{n\to\infty} \frac{3^n + 5^n}{2^n + 6^n}$

(iv) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \left(\frac{j}{n}\right)^3$ (hint: Riemann sums).

PM 2. Let $F(x) = \int_x^{x^2} e^{\sin(t)}\,dt$. What is $F'(x)$?

PM 3. For what real values of k do the vectors (3 k;

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PART I

AM 1. Let $\{a_n\}$ be the sequence defined recursively by

$$a_1 = 1,$$
$$a_{n+1} = a_n + n \cdot n! \quad \text{for } n \geq 1.$$

Compute a few values of a_n until you can guess a general formula for a_n, then prove that your guess is correct.

AM 2. For each of the following, either find the limit or explain divergence. (Here i is the usual number $\sqrt{-1}$.)

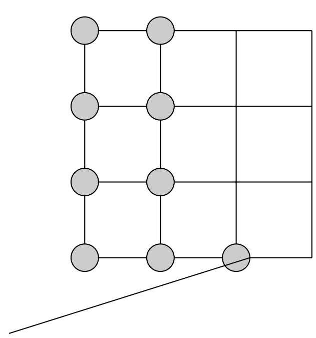
(a) $\displaystyle \lim_{n\to\infty} \frac{(2+\frac{i}{n})^2 - 4}{(3+\frac{i}{n})^2 - 9}$

(b) $\displaystyle \lim_{n\to\infty} \frac{4^{n+1} + (3i)^n}{4^{n+2} + (2i)^n}$

(c) $\displaystyle \lim_{n\to\infty} \sum_{j=0}^{\infty} \left(\frac{(2n+1)}{2n+3}\right)^j$

(d) $\displaystyle \sum_{n=0}^{\infty} \cos^3 \frac{n}{7}.$

AM 3. Suppose that positively and negatively charged particles are arranged in an $m \times n$ grid of the type shown here (in the $m = n = 4$ case).



randomly so that every node gets a particle. What is the expected number of attracting pairs in the grid?

AM 4. Consider the real matrices

$$A = \begin{bmatrix} 3 & 4 & 1 & 4 & 7 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 \\ 3 & 2 & 1 & 4 & 9 \end{bmatrix}; \qquad B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}:$$

You may take for granted that A and B are row equivalent.

(a) Find a basis of the row space of A.

(b) Find a basis of the column space of A.

(c) Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation whose matrix relative to the standard bases is A. Find a basis for the null space (kernel) of T.

(d) Find a basis of the image of T.

AM 5. Let c be a nonzero real constant. Consider the surface in $\mathbb{R}^3$,

$$S = \{(x; y; z) \in \mathbb{R}^3 : xyz = c\}:$$

Let $p = (p_1; p_2; p_3) \in S$, and let T be the tangent plane to S at p. Let the points of intersection of T with the three axes of $\mathbb{R}^3$ be $(u; 0; 0)$, $(0; v; 0)$, and $(0; 0; w)$. Show that the product uvw is independent of the point p. As part of your argument, explain why u, v, and w exist, i.e., why T actually intersects each axis.

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PART II

PM 1. Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

(a) For general $x \neq 0$, does $f'(x)$ exist? If so, what is it?

(b) Does $f'(0)$ exist? If so, what is it?

(c) Does $\lim_{x \to 0} f'(x)$ exist? If so, what is it?

(d) Is f' continuous at 0?

(As always, remember to explain your reasoning.)

PM 2. Let

$$A_1 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}; \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Show that one of the matrices A_i is diagonalizable over $\mathbb{R}$, and the other one is not. For the one which is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

PM 3. Integrate the function $f(x; y) = ye^{(x-1)}$